Circle Theorem – GCSE Maths

CONTENTS

1. Introduction

- Circle Theorems are a set of rules and properties related to angles, chords, and segments in a circle.
- They describe relationships between different geometric elements within and around a circle.

2. What are circle theorems?

- Circle theorems are special rules in geometry that describe relationships between angles, lines, and arcs in a circle.
- They help us find unknown angles or lengths using properties like angles in a semicircle, angles at the centre, and cyclic quadrilaterals, without the use of a protractor.
- This has very useful applications in engineering and design for analyzing circular patterns and structures.
- There are seven main circle theorems.

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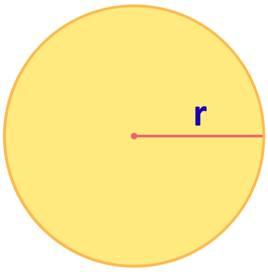
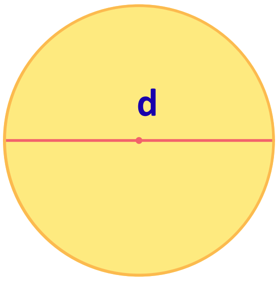
Basic Terminology of a Circle:

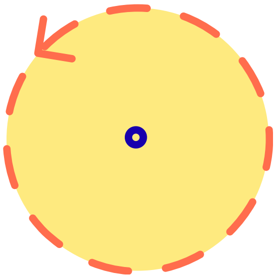
- **Radius(r):** A line from the center of the circle to any point on its edge.



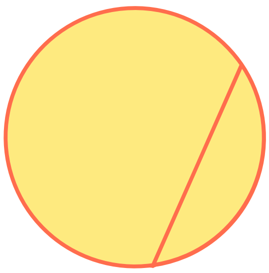
- **Diameter(d):** A line passing through the center, touching two points on the circle, equal to twice the radius.



- **Circumference:** The total distance around the circle.



- **Chord:** A line joining any two points on the circle but does not have to pass through the center.



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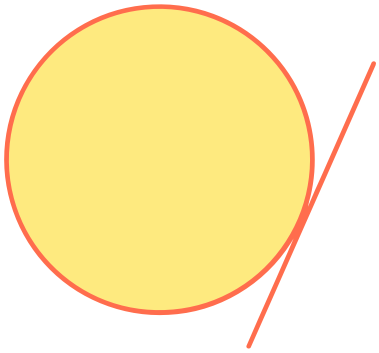
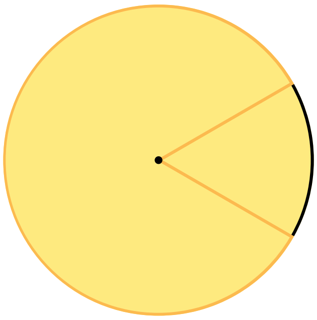
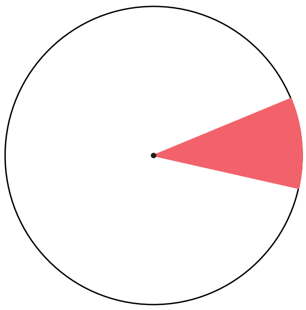
- **Tangent:** A line that touches the circle at exactly one point and does not cross it.



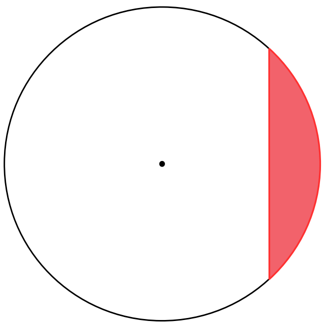
- **Arc:** A part of the circumference between two points.



- **Sector:** A part of a circle between two radii and the arc.



- **Segment:** The area between a chord and the arc above it.



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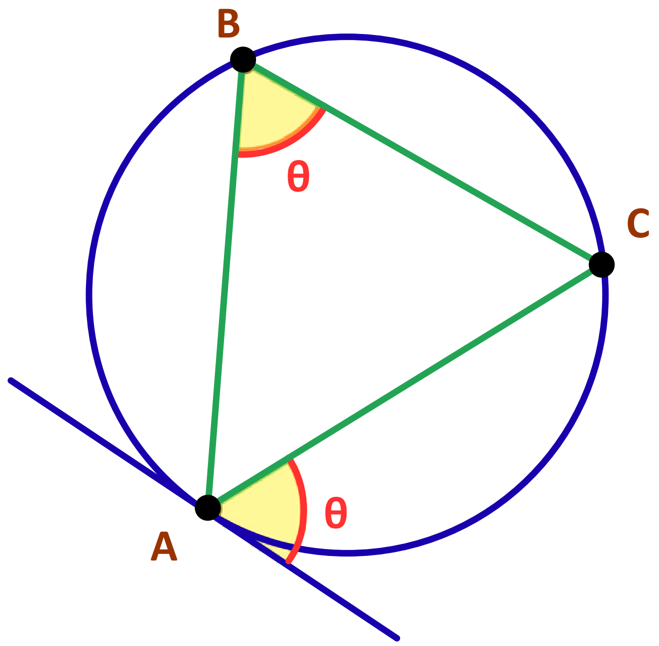
3. Circle Theorem 1: The alternate segment

- The angle that lies between a tangent and a chord is the same as the angle in the opposite part of the circle.
- It helps to find unknown angles in circle problems easily when tangents and chords are involved in geometry questions.



Steps to use the alternate segment theorem:

Step#1: Find and mark the important parts on the circle

Step#2: Use other angle rules to find one of the angles.

Step#3: Use the alternate segment theorem to find the other missing angle easily.

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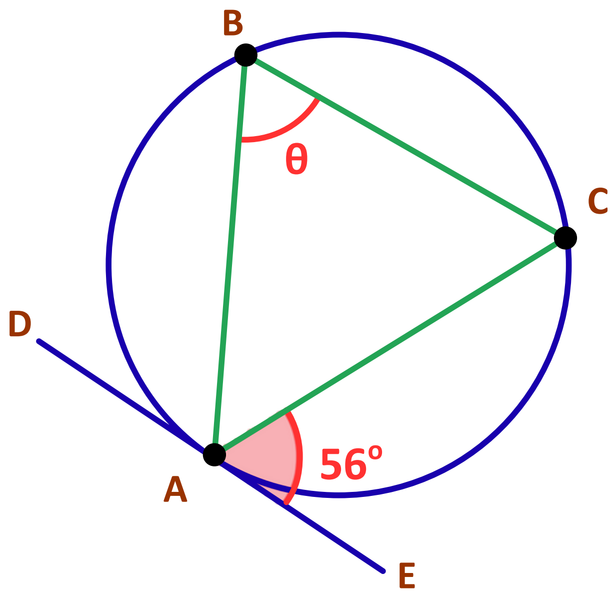
Example: Triangle ABC is inscribed in a circle with centre O. A tangent DE touches the circle at point A. If the angle CAE = 56°, calculate the size of the angle ABC.



Solution:

Step#1: Find and mark the important parts on the circle

Given:

- The tangent DE touching the circle at A.
- The chord AC meeting the tangent at A.
- The angle CAE = 56° (angle between the tangent and chord).

Step#2: Use other angle rules to find one of the angles.

Since we already know,

$$\angle CAE = 56°$$

No additional angle facts are needed for this step.

Step#3: Use the alternate segment theorem.

The Alternate Segment Theorem directly tells us that the angle between the tangent and the chord is equal to the angle in the opposite segment.

Thus, $\angle ABC = \angle CAE = 56°$

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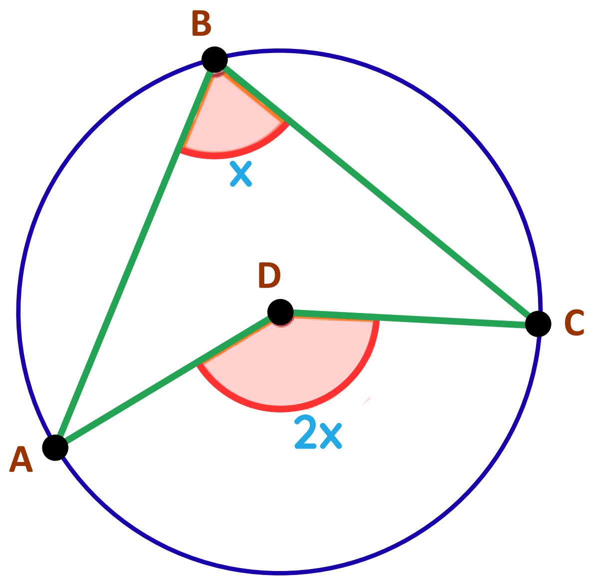
4. Circle Theorem 2: Angles at the centre and at the circumference

- The angle at the centre of a circle is twice the angle at the circumference when both angles stand on the same arc.
- It helps to find unknown angles in circle geometry problems when we know one of the two angles.



Steps to use the angle at the centre theorem:

Step#1: Find and mark the important parts on the circle

Step#2: Use other angle rules we know to find the angle at the centre or the angle at the edge (circumference).

Step#3: Use the angle at the centre theorem to find the missing angle

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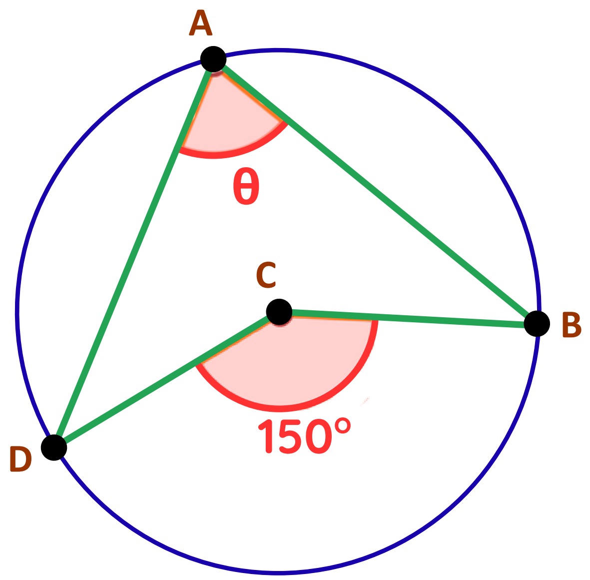
Example: In a circle with centre C, A, B, and D lie on the circumference, and if **∠BCD = 150°**, find ∠BAD.



Solution:

Step#1: Find and mark the important parts on the circle

Given:
- Angle at centre ∠BCD = 150°
- Angle at circumference ∠BAD = θ on the same arc.
- We have radius BC and DC.
- AB and AD are chords.

Step#2: Use other angle rules.

Since we already know,

∠BCD = 150°

No additional angle facts are needed for this step.

Step#3: Use the angle at the centre theorem to find the missing angle

Since the angle at the center is twice the angle at the circumference, we divide the given central angle by 2 to find ∠BAD.

BAD = 150 ÷ 2 = 75°

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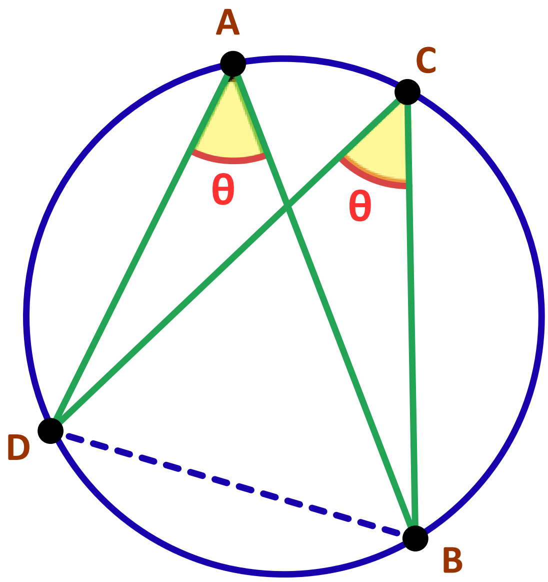
5. Circle Theorem 3: Angles in the same segment

- Angles in the same segment of a circle are equal.
- If we draw two angles on the circumference standing on the same chord, they will be equal, no matter where they are on that arc.
- It helps us to find unknown angles in circle geometry problems when angles stand on the same chord.



Steps to use the angles in the same segment theorem:

Step#1: Find and mark the important parts on the circle.

Step#2: Use any known angle rules to find one of the angles on the circumference in that segment.

Step#3: Use the angles in the same segment theorem to find the other angle (it will be equal).

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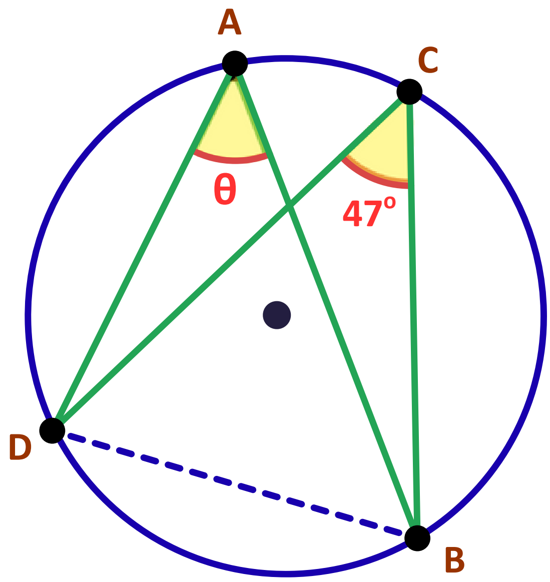
Example: In the circle below with centre O, if ∠DBC = 47°, calculate the size of ∠CAD.



Solution:

Step#1: Find and mark the important parts on the circle.

Given:
- The angle CBD = 47°
- AC and BD are chords

Step#2: Use any known angle rules to find one of the angles on the circumference in that segment.

Since we already know,

$$\angle DBC = 47°$$

No additional angle facts are needed for this step.

Step#3: Use the angles in the same segment theorem to find the other angle (it will be equal).

Using the Circle Theorem (angles in the same segment are equal):

Thus, $\angle CAD = \angle DAC = 47°$

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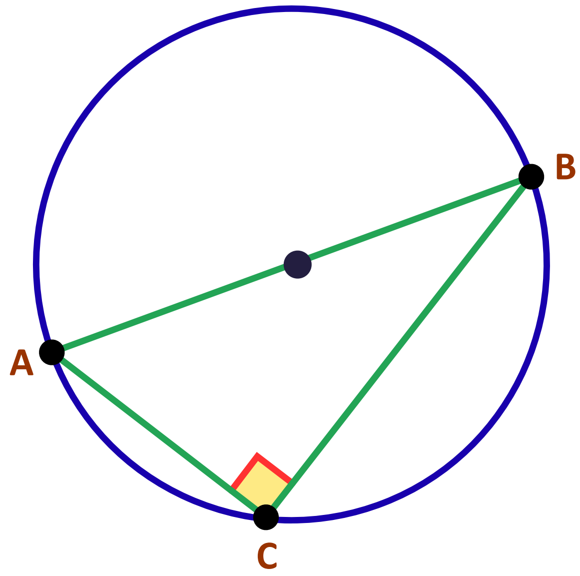
6. Circle Theorem 4: Angles in a semicircle

- The angle in a semicircle is always 90°.
- If we draw a triangle using the diameter of a circle, then the angle opposite the diameter will always be 90° or right angle.



Steps to use the angles in a semicircle theorem:

Step#1: Find and mark the diameter and the triangle on the circle.

Step#2: Use known angle facts to find any other needed angles in the triangle if required.

Step#3: Use the semicircle theorem to state that the angle opposite the diameter is 90°.

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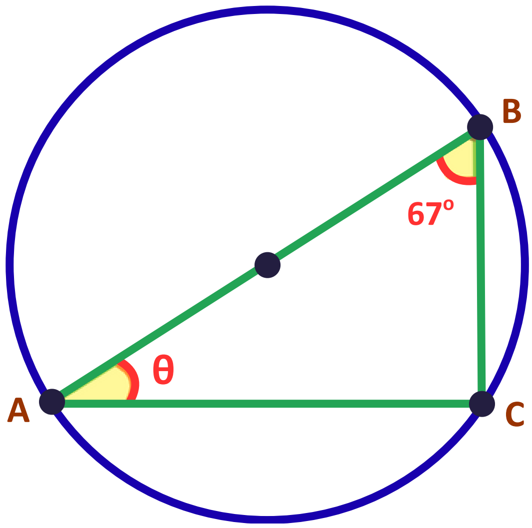
Example: In a circle, ABC is a triangle with AB as the diameter and ∠ABC = 58°. Find ∠BAC.



Solution:

Step#1: Find and mark the diameter and the triangle on the circle.

Given:
- AB is the diameter.
- △ABC lies on the circle.

Step#2: Use known angle facts

Sum of angles in a triangle:

$$\theta + \angle ABC + \angle ACB = 180^\circ$$

Step#3: Use the semicircle theorem

As the angle in a semicircle is equal to 90°, so

$$\angle ACB = 90^\circ$$

Then, sum of angles:

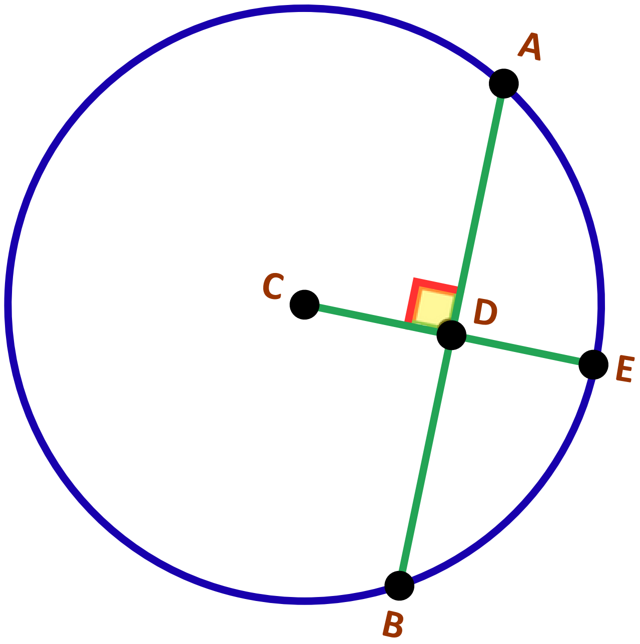
$$\theta + 67^\circ + 90^\circ = 180^\circ$$

$$\theta = 180^\circ - 157^\circ$$

$$\theta = 23^\circ$$

7. Circle Theorem 5: Chord of a circle

- When we draw a perpendicular line from the center of a circle to any chord, it neatly splits that chord into two equal parts.
- It helps us to find unknown lengths in geometry problems and proves equal parts on either side of the chord.



Steps to find missing lengths using chords:

Step#1: Mark the important parts (centre, chord, and the perpendicular from the centre to the chord).

Step#2: Use any known angle rules if we need to find missing angles in the triangle formed.

Step#3: Use Pythagoras' theorem or trigonometry to find the missing length.

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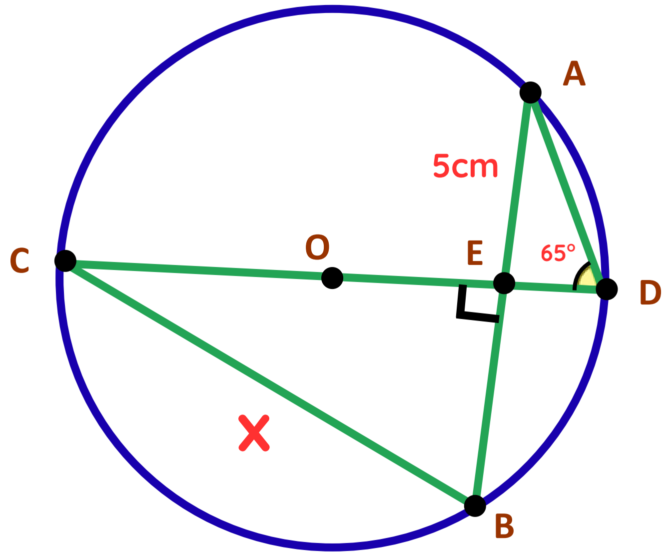
Example: Calculate the length of chord BC, given that **AE = 5 cm, ∠ADE = 65°**, and **AB ⊥ CD** at E, with O as the centre of the circle.



Solution:

Step#1: Find and mark the diameter and the triangle on the circle.

Given:
- O is the centre of the circle.
- Chord BC is perpendicularly bisected by OE (since AB⊥CD at E, and O is the centre).
- AE = 5 cm, ∠ADE=65°

Step#2: Use any known angle rules if we need to find missing angles in the triangle formed.

- **Angles:**

  ∠ABC = ∠ADE = 65° (angles in the same segment are equal).

- **Lengths:**

  Since the centre line BE is perpendicular to chord AD, it splits it evenly. So, BE = AE = 5 cm.

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Step#3: Find Radius OB

Using △ABE:

$$\text{Cos}(65^{\circ}) = \frac{AE}{AB}$$

$$AB = \frac{5}{0.4226}$$

$$AB = 11.83 \text{ cm}$$

Step#4: Find Chord BC

- Find half-chord (BE):

$$BE = \sqrt{(11.8)^2 - 5^2}$$

$$BE = \sqrt{115}$$

$$BE = 10.7 \text{ cm}$$

- Double it for full chord (BC):

$$BC = 2 \times 10.7 \text{ cm}$$

$$BC = 21.4 \text{ cm}$$

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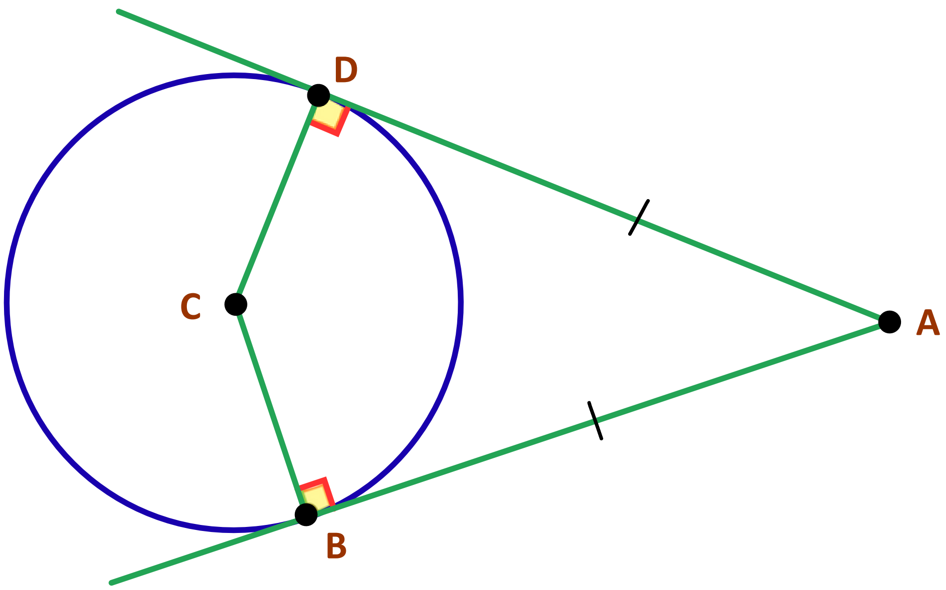
8. Circle Theorem 6: Tangent of a circle

- At the point where a tangent touches a circle, it forms a right angle (90°) with the radius drawn to that point.
- This theorem helps calculate unknown angles and verify right angles in circle geometry problems.



Steps to use the tangent of a circle theorems:

Step#1: Mark the important parts.

Step#2: Use any other angle facts you know to find missing angles near the tangent.

Step#3: Use the tangent theorem to find the missing angle.

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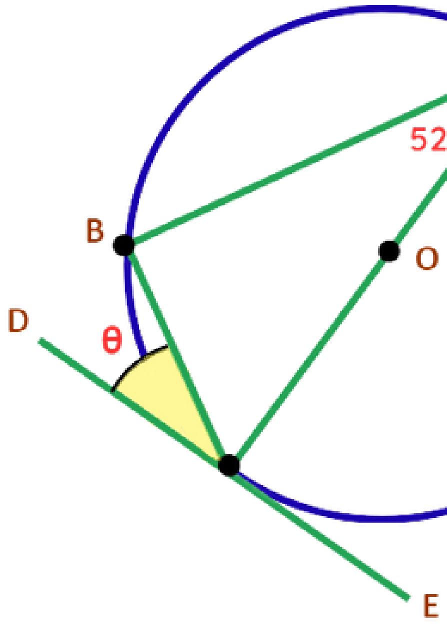
Example: Points A, B, and C lie on the circumference of a circle with centre O. Line DE is a tangent at point AA. If angle ACB = 63° , find angle BAD.



Solution:

Step#1: Mark the important parts.

Given:
- DE is a tangent to the circle at point A.
- AC is a chord that meets the tangent.
- ∠BAD = θ is the angle in the alternate segment.
- ∠ACB = 63°

Step#2: Use any other angle facts you know to find missing angles near the tangent.

∠ACB = 63° is on the opposite side of chord AC from the tangent.

Step#3: Use the tangent theorem.

Using the Alternate Segment Theorem, the angle between the tangent and the chord equals the angle in the alternate segment. So:

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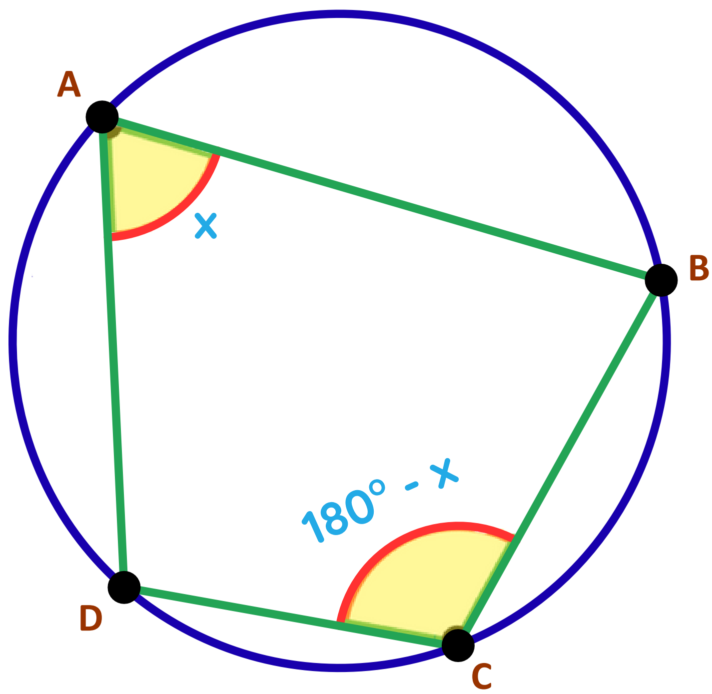
9. Circle Theorem 7: Cyclic quadrilateral

- In a quadrilateral with all corners on the circle, the opposite angles add up to 180°.
- If a 4-sided shape is inside a circle, then:

Opposite angles = 180°



Steps to use the cyclic quadrilateral theorem:

Step#1: Mark the key parts.

Step#2: Use any angle rules you know to find one of the opposite angles in the quadrilateral.

Step#3: Use the cyclic quadrilateral theorem to find the other missing angle.

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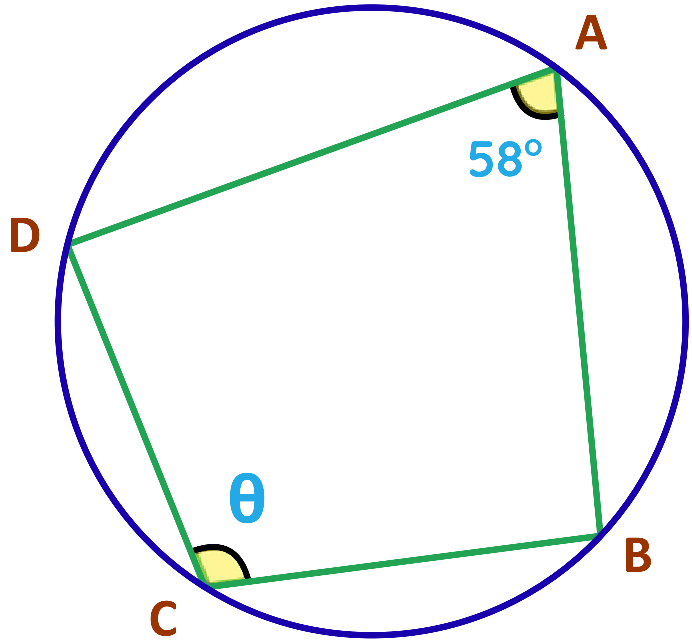
Example: ABCD is a cyclic quadrilateral where A, B, C, and D lie on the circumference of a circle. If angle DAB = 58°, calculate the size of angle BCD.



Solution:

Step#1: Mark the key parts.

Given:

- The angle BAD = 51°
- The angle BCD = θ

Step#2: Use any angle rules you know to find one of the opposite angles in the quadrilateral.

- This is a cyclic quadrilateral, so opposite angles add up to 180°.
- In cyclic quadrilaterals:

$$\angle DAB + \angle BCD = 180^\circ$$

Step#3: Use the cyclic quadrilateral theorem to find the other missing angle.

$$\angle BCD = 180^\circ - \angle DAB$$
$$\angle BCD = 180^\circ - 58^\circ$$
$$\angle BCD = 122^\circ$$

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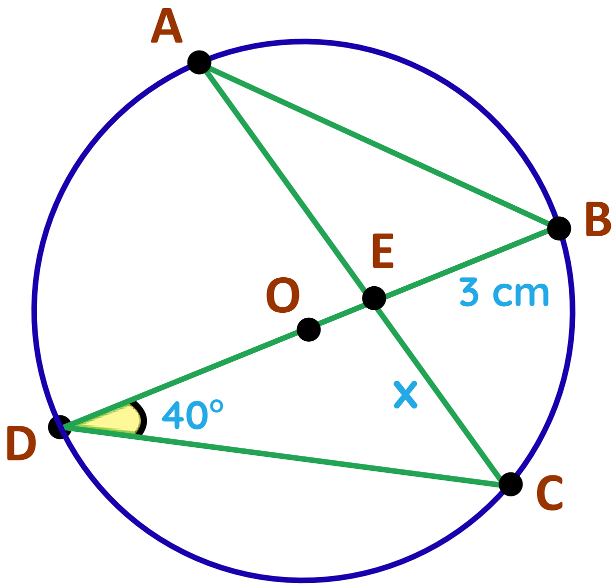
10. Solved Examples

Problem1: Points A, B, C, and D lie on a circle with centre O. BD is the diameter, and AC is a chord perpendicular to the diameter at point E. If BE = 3 cm and ∠CDE = 40°, calculate the distance x, which is the length from C to E.



Solution:

Step#1: Mark the key parts.

Given:
- CE is perpendicular to BD (right angle at E)
- Triangle CDE is right-angled at E
- BE = 3 cm, ∠CDE = 40°

Step#2: Use angle facts

In triangle CDE:

- ∠CED = 90° (since CE ⊥ BD)
- ∠CDE = 40° (given)
- Use angle sum in triangle:

$$\angle DCE = 180^\circ - 90^\circ - 40^\circ$$

$$\angle DCE = 50^\circ$$

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Step#3: Use tan to find x

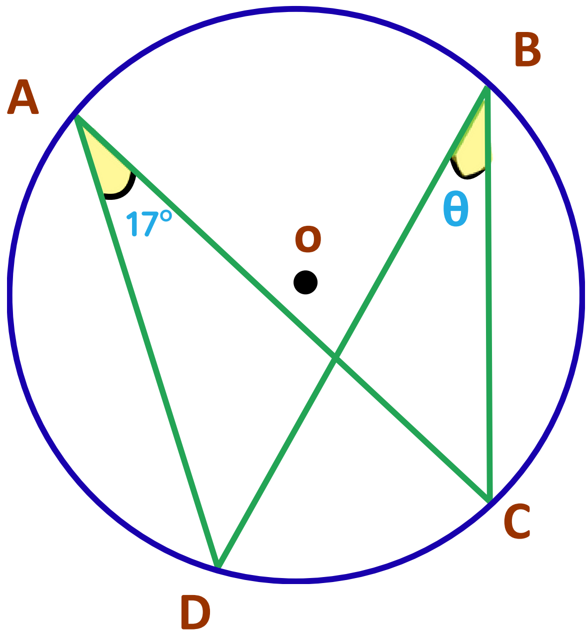
$$\text{Tan}(40°) = \frac{x}{3}$$

$$x = 3 \times \text{Tan}(40°)$$

$$x = 3 \times 0.8391$$

$$x = 2.52\text{cm}$$

Problem2: A circle with centre O has four points on the circumference: A, B, C, and D. Angle ∠CAD = 17°. Find the size of angle ∠CBD.



Solution:

Step#1: Understand the Figure

∠CAD and ∠CBD are angles subtended by the same chord CD on opposite sides of the circle.

Step#2: Apply the Circle Theorem

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Angles in the same segment are equal.

That means:

$$\angle CAD = \angle CBD$$

Step#3: Conclude the answer

Since $\angle CAD = 17°$,

Then:

$$\angle CBD = 17°$$